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# Methodological Issues of the Fuzzy Set Theory (Generalizing Article)

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**Abstract**—The theory of fuzziness is an important area of modern theoretical and applied mathematics. The methodology of the theory of fuzziness is a doctrine of organizing activities in the field of development and application of the scientific results of this theory. We discuss some methodological issues of the theory of fuzziness, i.e., individual components of the methodology in the area under consideration. The theory of fuzziness is a science of pragmatic (fuzzy) numbers and sets. Ancient Greek philosopher Eubulides showed that the concepts of "Heap" and "Bald" cannot be described using natural numbers. E. Borel proposed to define a fuzzy set using a membership function. A fundamentally important step was taken by L.A. Zadeh in 1965. He gave the basic definitions of the algebra of fuzzy sets and introduced the operations of intersection, product, union, sum, and negation of fuzzy sets. The main thing he did was demonstration of the possibilities of expanding ("doubling") mathematics: by replacing the numbers and sets used in mathematics with their fuzzy counterparts, we obtain new mathematical formulations. In the statistics of nonnumerical data, methods of statistical analysis of fuzzy sets have been developed. Specific types of membership functions are often used— interval and triangular fuzzy numbers. The theory of fuzzy sets in a certain sense is reduced to the theory of random sets. We think fuzzy and that is the only reason we understand each other. The paradox of the fuzzy theory is that it is impossible to consistently implement the thesis "everything in the world is fuzzy." For ordinary fuzzy sets, the argument and values of the membership function are crisp. If they are replaced by fuzzy analogs, then their description will require their own clear arguments and membership functions, and so on ad infinitum. System fuzzy interval mathematics proceeds from the need to take into account the fuzziness of the initial data and the prerequisites of the mathematical model. One of the options for its practical implementation is an automated system-cognitive analysis and Eidos intellectual system.

**Keywords:** mathematical research methods, theory of fuzzy sets, methodology, "Heap" paradox, membership function, interval mathematics, triangular fuzzy numbers, random sets, fuzzy theory paradox, system fuzzy interval mathematics

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## INTRODUCTION

The theory of fuzziness is a field of modern theoretical and applied mathematics. A number of mathematical research methods considered in our journal are based on its application [1–6].

Not only individual models, methods, and algorithms but also the entire theory of fuzziness, including its formation and development, deserves discussion. This is due, in particular, to the fact that inadequate ideas about this area of mathematics are widespread. We include the theory of fuzziness in systemic fuzzy interval mathematics [5]. Fuzzy sets are objects of nonnumerical nature, which are studied by statistics of objects of nonnumerical nature (statistics of nonnumerical data, nonnumerical statistics). The development and content of this scientific field have been repeatedly discussed in our journal.

Methodology is the study of the organization of activity ([7], p. 20); therefore, the methodology of the theory of fuzziness is the study of the organization of activity in the field of development and application of scientific results of this theory. This paper considers some methodological issues of the theory of fuzziness, i.e., individual components of the methodology in the area under consideration. The basic concepts of fuzzy set theory are assumed to be known. They are given in many publications, including the author's textbooks on applied statistics, decision theory, econometrics, and organizational and economic modeling, and in monographs on systemic fuzzy interval mathematics. However, the methodological issues considered below have not previously been discussed in detail.

## THEORY OF FUZZINESS IS A SCIENCE OF PRAGMATIC NUMBERS AND SETS

In short, classical mathematics is a science of numbers and figures (sets) [8].

However, the properties of numbers used in practical life (they are called pragmatic) quite often differ from the properties of mathematical numbers. For example, pragmatic numbers have only a few significant digits, and their number is finite, while the set of real numbers has the power of a continuum. This does not mean that it is necessary to abandon the intellectual tools developed by mathematicians over millennia. It is advisable to assume that pragmatic numbers are mathematical numbers with errors. The boundaries of sets used in practice are also fuzzy. This is the approach being developed in systemic fuzzy interval mathematics (see [5] and references to monographs in the list of References in this publication). In particular, what is stated above makes clear the need to study the stability of conclusions obtained in mathematical models in relation to permissible deviations of the initial data and assumptions of the models.

The theory of fuzziness makes it possible to develop mathematical tools for working with pragmatic numbers and sets. Often, the starting point is the work of L.A. Zadeh [9], in which the term fuzzy set appeared. In the domestic tradition, set is translated as a multitude, but for fuzzy, there is a range of terms—washed-out, indistinct, vague, foggy, fluffy. We will use the term fuzzy set and call the corresponding area of mathematics "fuzzy set theory" or "theory of fuzziness."

The founder of the theory of fuzziness, L.A. Zadeh (1921–2017), considered the theory of fuzzy sets primarily as an apparatus for analyzing and modeling humanistic systems, i.e., systems in which humans participate. His approach is based on the premise that "the elements of human thinking are not numbers, but elements of some fuzzy sets or classes of objects for which the transition from "membership" to "nonmembership" is not abrupt, but continuous" [10]. The theory of fuzziness has also proven useful for analyzing and modeling technical systems [1–3], for risk assessment [6], in economics [4], and in management [11], i.e., in virtually all areas of science and practice. Tens of thousands of studies and numerous journals are devoted to this theory. The journal *Fuzzy Systems and Soft Computing* is published in our country.

## DEVELOPMENT OF THEORY OF FUZZINESS

The first monograph by a Russian author on theory of fuzziness is the book [12], published in 1980. It summarizes the work of the 1970s; in particular, it is established that the origins of the theory of fuzziness date back to the times of Ancient Greece.

The philosopher Eubulides (fourth century BC) discussed two paradoxes (aporia, sophisms): "Heap" and "Bald." The first of them is: "One grain is not a heap. If you add grain to grain, at what point does a heap appear?" The second is: "Having lost one hair, you do not yet become bald; having lost a second hair, you are the same; when does baldness begin?" [13]. These paradoxes are close to the famous aporias of Zeno of Elea. They were analyzed by many philosophers, including Hegel [14]. These paradoxes show that the concepts of "Heap" and "Bald" cannot be described using natural numbers. It is impossible to specify a natural number $N$ such that a set of $n$ grains with $n < N$ is not a heap, and with $n \geq N$, it is already recognized as a heap. Obviously, the boundary between "Heap" and "Not a heap" is blurred. In other words, to describe the linguistic variable "Heap" that takes on two values, "This is a heap" and "This is not a heap," we should use not the theory of natural numbers, but a more complex mathematical construction. Currently, such a construction is the theory of fuzzy sets. In the works of the 1970s, we used the "Heap" paradox as one of the justifications for the need to develop and use the theory of fuzzy sets (for details and references, see [12]). Later, other authors also made the same arguments (see, for example, the monograph [15]). The French mathematician E. Borel (1871–1956), who, together with R. Baer and A. Lebesgue, was one of the founders of the theory of measure and its applications in probability theory, proposed to describe a fuzzy set using a membership function. In the monograph *Probability and Certainty* [16], published in French in 1956, i.e., nine years before the publication of L.A. Zadeh's seminal work [9], Borel proposed describing the concept of "Heap" using the function $f(n)$ (from the natural argument $n$) defined as the proportion of people calling a set of $n$ grains "Heap" among all speakers of French (in Russian translation for Russians). Thus, E. Borel not only introduced a new mathematical tool for describing a linguistic variable but also indicated a method for estimating the membership function on the basis of experimental data (according to the results of a survey) [12]. The methodology for applying this evaluation method is disclosed in our textbooks. For example, in marketing, when studying the market, the concept of "rich" is used. To describe this linguistic variable, as well as the linguistic variable "Heap," it is advisable to use a membership function, the estimate of which can be obtained by asking respondents a series of questions like "Do you think that a person with a monthly income of $N$ rubles is rich?" In a similar way, one can describe various linguistic variables whose values are words of natural language, for example, "small–medium–large." The use of linguistic variables and corresponding membership functions for risk assessment is considered in [6].

A fundamentally important step in the development of theory of fuzziness was made by L.A. Zadeh in 1965 [9]. He gave the basic definitions of the algebra of fuzzy sets by introducing the operations of intersection, product, union, sum, and negation of fuzzy sets. Mathematical problems immediately arose—a formula for the algebra of (ordinary) sets is given (for example, one of De Morgan's laws); is it true in theory of fuzziness? It is important that the analogs of the intersection operation in the algebra of (ordinary) sets in theory of fuzziness are two operations—intersections and products. Depending on which of these analogs is used, the formula for the algebra of sets in theory of fuzziness may be either true or false.

The main thing is that L.A. Zadeh demonstrated the possibilities of expanding mathematics, which are important for theory and practice. Thus, by replacing the numbers and sets used in mathematics with their fuzzy analogs, we obtain new mathematical formulations (this approach was considered above for the algebra of sets). New formulations can be studied theoretically, or they can be applied in solving various practical problems. To implement this scientific program, a movement of researchers of fuzziness developed, the recognized leader of which was L.A. Zadeh. He himself actively carried out large scientific and organizational work, speaking at conferences, attracting and stimulating new researchers of fuzziness problems. As already noted, in the decades since 1965, the theory of fuzziness has become a developed area of theoretical and applied mathematics.

One of the sections of mathematics is mathematical logic, devoted to the formalization and mathematical study of the provisions of logic—the science of the forms and laws of thought. As is known, Aristotle (384–322 BC) is the true creator of logic as a science. The beginning of mathematical logic is considered to be the works of George Boole (1815–1864). A section of mathematical logic is named after him: Boolean algebra (algebra of logic). It considers the rules for establishing the truth or falsity of complex statements on the basis of the truth or falsity of simple statements, on the basis of which complex ones are constructed. Boolean algebra corresponds to the algebra of sets. For example, the truth set of the statement "*A* and *B*" is the intersection of the truth sets of the statements *A* and *B*, and the truth set of the statement "*A* or *B*" is the union of the truth sets of the statements *A* and *B*. In classical logic, a statement is either true or false; i.e., its truth value takes only two values. However, it is quite natural to introduce into consideration a theory in which the truth value of a statement takes several values (for example, certainly true–most likely–possible–unlikely–certainly false). There may be an infinite number of truth values. Thus, we turn to fuzzy logic, which has been studied since the 1920s as infinite-valued logic, in particular, by J. Lukasiewicz (1878–1956) and A. Tarski (1901–1983). The paradoxes of Eubulides are used as one of the justifications for the need to consider fuzzy logic.

It is quite natural that the founder of the modern theory of fuzziness, L.A. Zadeh, turned to the theory of fuzzy logic in 1973. This led to unexpected results—some of his followers began to call the ENTIRE theory of fuzziness fuzzy logic. This is clearly absurd, since logic is a normative science about the laws, forms, and methods of intellectual activity. It relates to human thinking, while the theory of fuzziness can be used in any field of science and branch of the national economy. However, the use of the term "fuzzy logic" (instead of "theory of fuzziness" or "fuzzy set theory") is widespread, especially among those authors who are not sufficiently familiar with the history of theory of fuzziness.

## STATISTICAL ANALYSIS OF FUZZY DATA

In system fuzzy interval mathematics and the central area of statistics of nonnumerical data—statistics in spaces of arbitrary nature, methods for analyzing fuzzy data are developed.

How to estimate the membership function from sample data is described above. Our work on theory of fuzziness began with similar formulations in the theory of tolerances (reflexive symmetric binary relations) (1975). The estimation of fuzzy tolerance was carried out on the basis of the analysis of a sample whose elements were independent identically distributed random tolerances distributed in accordance with the estimated fuzzy tolerance.

Fuzzy data statistics uses distances (metrics) in spaces of fuzzy sets. Since fuzzy sets are a special type of objects of nonnumerical nature, then, when processing a sample whose elements are fuzzy sets, various methods of analyzing statistical data of arbitrary nature can be used by calculating averages, nonparametric estimates of density, restoring dependences (regression analysis), constructing diagnostic rules, etc.

Membership functions of a specific type are often used. Let us give examples. The simplest membership function takes the value 1 on a certain interval and the value 0 outside it. A fuzzy number is described by an interval. We come to interval mathematics, in which intervals are used instead of numbers. This area of mathematics is also called interval analysis, interval arithmetic, or interval calculations. The beginning of the development of interval mathematics, as well as the theory of fuzziness, dates back to the times of Ancient Greece. As early as the third century BC, Archimedes calculated the lower and upper bounds for the number "pi." In the second half of the 20th century, the needs of computer calculations caused the rapid development of interval analysis almost simultaneously and independently in the Soviet Union, the United States, Japan, and Poland. Statistics of fuzzy data includes statistics of interval data. It is described in detail in our textbooks on applied statistics and its central part (nonnumerical statistics) and on decision theory, and in monographs on system fuzzy interval mathematics. Works on statistics of interval data are regularly published in our journal (see, for example, [17–25]).

In interval data statistics, the values of the membership function change abruptly when the argument changes, from 0 to 1 and back from 1 to 0. However, although the numerical data are fuzzy, L.A. Zadeh's requirement "the transition from "membership" to "non-membership" is not abrupt, but continuous" is not met. Therefore, it is natural to use continuous membership functions to specify fuzzy numbers. Among the various types of fuzzy numbers, triangular fuzzy numbers stand out. They are specified by three real numbers: $a < b < c$. The membership function is equal to 0 when the argument value is $x < a$, linearly increases from 0 to 1 when $a \leq x < b$, linearly decreases from 1 to 0 when $b \leq x < c$, and is equal to 0 when $c \leq x$. For triangular fuzzy numbers, the main part of the graph of the membership function has a triangular shape, hence the name. Arithmetic operations on triangular fuzzy numbers are defined in [6]. It is important that the results of addition, subtraction, multiplication, and division of triangular fuzzy numbers are always triangular fuzzy numbers; in other words, they remain in the three-parameter set of triangular fuzzy numbers. This property significantly simplifies calculations compared to schemes using membership functions of arbitrary type. An example is risk assessment using an additive-multiplicative model [6].

Other membership functions are also used. For example, a four-parameter family of membership functions, the graphs of the membership functions of which have the form of a trapezoid. They are defined by four real parameters: $a < b < c < d$. The membership function is equal to 0 for the argument value $x < a$, linearly increases from 0 to 1 for $a \leq x < b$, is equal to 1 for $b \leq x < c$, linearly decreases from 1 to 0 for $c < x < d$, and is equal to 0 for $x > d$. Such a trapezoidal membership function has the main features of a general membership function commonly used in applied problems.

## THEORY OF FUZZINESS AND PROBABILITY THEORY

Since the emergence of modern theory of fuzziness in the 1960s, its relationship to probability theory has been discussed. The fact is that the membership function of a fuzzy set resembles a probability distribution density. The only difference is that the sum of the probabilities over all possible values of a random variable (or the integral, if the set of possible values is uncountable) is always equal to 1, and the sum $S$ of the membership function values (in the continuous case, the integral of the membership function) can be any nonnegative number. There is a temptation to normalize the membership function, i.e., to divide all its values by $S$ (for $S \neq 0$), in order to reduce it to a probability distribution (or probability density). However, fuzziness specialists rightly object to such a "primitive" reduction, since it is carried out separately for each fuzziness (fuzzy set), and the definitions of ordinary operations on fuzzy sets cannot be reconciled with it. The last statement means the following. Let the membership functions of fuzzy sets $A$ and $B$ be transformed in the manner indicated. How are the membership functions $A \cap B$, $A \cup B$, $A + B$, and $AB$ transformed? It is impossible to establish this in principle. The last statement becomes completely clear after considering several examples of pairs of fuzzy sets with the same sums of the values of the membership functions, but different results of set-theoretic operations on them. Moreover, the sums of the values of the corresponding membership functions for these results of set-theoretic operations, for example, for the intersections of sets, are also different. In works on fuzzy sets, it is sometimes asserted that the theory of fuzziness is an independent section of applied mathematics and has no relation to probability theory. Some authors who discussed the relationship between the theory of fuzziness and probability theory emphasized the difference between these areas of theoretical and applied research. Usually, axiomatics are compared and areas of application are compared.

Arguments in the second type of comparisons do not have the force of proof, since there are different opinions regarding the limits of applicability of even such a long-standing scientific field as probabilistic-statistical methods. Moreover, there is no consensus on arithmetic. Let us recall the result of the reasoning of one of the most famous French mathematicians, Henri Lebesgue, regarding the limits of applicability of arithmetic: "Arithmetic is applicable when it is applicable" ([26], pp. 21, 22).

When comparing different axiomatics of the theory of fuzziness and probability theory, it is easy to see that the lists of axioms differ. This does not mean that a connection cannot be established between the indicated theories, for example, like the well-known reduction of Euclidean geometry on a plane to arithmetic, or more precisely, to the theory of the $R^2$ number system [27]. These two axiomatics (Euclidean geometry and arithmetic) at first glance, differ greatly.

One can understand the desire of the enthusiasts of the theory of fuzziness to emphasize the fundamental novelty of their scientific apparatus. However, it is no less important to establish the connections of this approach with previously known ones. As it turned out, the theory of fuzzy sets is closely related to probability theory; in a certain sense, it is reduced to the theory of random sets and is included in it.

As early as 1975, it was shown that fuzzy sets are naturally considered as "projections" of random sets [12]. A function equal to the probability of covering an element with a random set $A$ can be considered as a membership function of some fuzzy set $B$. In this case, one say that $B$ is a projection of $A$. The basic theorem is that, for any fuzzy set $B$, one can construct a random set $A$ such that $B$ is a projection of $A$.

The purpose of reducing the theory of fuzzy sets to the theory of random sets is to see behind any construction of fuzzy sets a construction of random sets that determines the properties of the former. In this case, the projections of fuzzy sets and the result of operations on them are the corresponding fuzzy sets and the results of operations on them. Fuzzy sets can be seen as random sets in the same way as we see a random variable behind the probability density function. In particular, necessary and sufficient conditions have been found under which the projection of the intersection of random sets yields the product or intersection of fuzzy sets.

In the early 1980s, similar approaches began to develop abroad. One of the works has a remarkable title, "Fuzzy Sets as Equivalence Classes of Random Sets" [28].

## WE THINK FUZZY AND THAT IS THE ONLY REASON WE UNDERSTAND EACH OTHER

New scientific results can be obtained when performing specific applied work or when preparing general publications (including formally popular science publications). Thus, when developing GOST 11.011-83 "Applied Statistics. Rules for Determining Estimates and Confidence Limits for Gamma Distribution Parameters," the basic ideas of interval data statistics were formulated and applied, and one-step estimates of parameters of distributions were proposed, which have advantages over maximum likelihood estimates, which are still traditionally included in textbooks on probability theory and mathematical statistics. However, GOST 11.011-83 as the basis for new scientific results was canceled in 1987 along with the entire series of standards "Applied Statistics," removed from libraries, and is currently poorly accessible.

The first book by a Russian author on fuzzy sets [12] was published in the popular science series "Mathematics. Cybernetics" by the Znanie Publishing House. It was a scientific monograph summarizing the author's work on the topic under consideration in the 1970s. After its publication, the editors of the journal *Science and Life* suggested publishing an article on the theory of fuzziness, which was done [29]. In this article, along with the fundamentals of the theory of fuzziness, a number of methodological problems of this theory were considered. Because of the status of popular science works, the scientific community rarely referred to these publications, although they picked up and began to develop the ideas put forward in them.

One of the central ideas of the article [29] can be briefly formulated as follows: "We think fuzzy and that is why we can understand each other." This statement is related to problems of terminology. Many authors rightly demand that the terms used be defined. However, there are several problems associated with the implementation of this requirement.

First, a specific term is defined with the help of other terms, and they, in turn, should also be defined. Obviously, sooner or later we come to basic terms that can no longer be defined. In the same way, in mathematics, revealing concepts and rules of reasoning, we come to axioms. What is the basis for the confidence that basic concepts and axioms are understood by everyone in the same way? In real research, the saving grace is that they get to basic concepts and axioms extremely rarely.

Secondly, the authors of numerous publications provide different definitions. A natural desire arises to compare and contrast them. However, such activity is labor-intensive, there is a danger of falling into the captivity of modern versions of scholasticism and moving away from solving real problems and practically important tasks. Eubulides already understood the importance of the problem under consideration (see above). People usually do not have problems using the concepts of "heap" and "bald," despite the vagueness of these concepts, which is especially evident when translating from one language to another. Problems arise when building and using artificial intelligence systems [5].

## PARADOX OF THE THEORY OF FUZZINESS

The authors of [29] discuss the main paradox of fuzziness. Let us consider it. The concept of fuzziness has its own approach to understanding the world and to constructing models of real phenomena. According to it, everyone strives to see fuzziness in everything and to model this fuzziness with the help of suitable fuzzy objects.

Many theoretical and applied works based on the theory of fuzziness have convincingly demonstrated that this approach is reasonable and useful. There is a temptation to proclaim the thesis: "Everything in the world is fuzzy." It looks especially attractive owing to the great harm of deceptive clarity. But can this thesis be carried out consistently?

A fuzzy set is defined by its membership function. Let us pay attention to the argument and the meaning of this function. Are these objects clear or fuzzy? The thesis "Everything in the world is fuzzy" suggests that they are fuzzy.

As an example, let us consider the sophism "Heap." First, let us consider the argument of the function—the number of grains relative to the totality of which the question is decided: "Is this a heap or not a heap?" Is it possible to know the number of grains in a sufficiently large totality absolutely precisely? No matter how you count the grains (manually, by weight, automatically) errors are always possible (a person can make a mistake, a machine can break down...). The situation is similar for many other examples of fuzzy sets.

Now let us discuss the meaning of the membership function. It is even more fuzzy! Does it make sense to express a person's opinion with at least three significant digits? It is generally recognized in sociology that a person in verbal assessments usually cannot distinguish more than three or at best six gradations. From this, using the appropriate calculation, we can find that the membership function expressing the opinion of one person can be determined only with an accuracy of 0.17 to 0.33. People's opinions should be represented not by ordinary graphs (thin lines), but by fairly wide stripes.

If the membership function is constructed as an average of individual opinions, then its values are by no means absolutely accurate owing to the fact that the surveyed population usually does not include even a small proportion of those who could be surveyed. Confidence limits for the results of sample surveys have been developed in applied statistics. With a number of respondents up to 100, the probability of a certain answer (one of two) in the general population can only be estimated with an accuracy of 10%.

And only if the values of the membership function are calculated using certain algorithms (for example, analytical formulas) are they known absolutely precisely. But then the question arises: How justified are these algorithms (formulas) themselves? It usually turns out that their justification is rather weak...

What is the result of the discussion? Both the argument and the value of the membership function should be considered fuzzy.

What follows from this? Let us start again with the argument. It itself is not a strictly defined value, but a fuzzy set of values, which means it is described by a certain membership function, and it is determined by the values of some of its arguments—a second-order argument. But this new argument is also fuzzy! Again, a membership function appears from some new argument—a third-order argument. And so on.

Will we ever stop on this path? If we stop, we will have to use crisp values of the argument, and this contradicts the thesis "Everything in the world is fuzzy." According to this thesis, crisp values are fictitious; nothing in the real world corresponds to them. If we do not stop, we will get an infinite sequence of fuzzy models, in which a new fuzziness crawls out of each fuzzy set, like from a Russian nesting doll.

We obtain a similar infinite sequence when considering the membership function of a fuzzy set. Let us introduce the appropriate notation. Fuzzy sets of the first type "work" with a fixed membership function, while for fuzzy sets of the second type, the membership function itself is fuzzy. If the fuzziness describing the membership function is itself foggy, then we are forced to consider fuzzy sets of the third type, etc.

Of course, the described paradox does not prevent the successful use of fuzzy mathematics in specific applications. It is usually believed that the values of the argument and the membership function are completely defined and crisp; i.e., fuzzy systems of the first type are used. In the article [30], L.A. Zadeh considered fuzzy sets of the second type and noted the need to study the fuzzy sets of the third type, fourth type, ..., *i*th type. However, by now, experience has been accumulated in studying and applying fuzzy sets of only the first type and, in some cases, the second type (see, for example, [31–34]). Fuzzy sets of higher types are hardly considered. At the same time, the values of the arguments are always considered crisp. It should be noted that only the first steps have been made in implementing the thesis "Everything in the world is fuzzy."

## CONCEPT OF SYSTEM FUZZY INTERVAL MATHEMATICS

Revolutionary changes are taking place in the field of mathematical research methods. A new paradigm of mathematical research methods is replacing the outdated paradigm of the mid-20th century. The new paradigm is based on system fuzzy interval mathematics and its important component of statistics of nonnumerical data, also known as "statistics of nonnumerical objects" and "nonnumerical statistics." Currently, most of the articles in our journal on applied statistics relate to statistics of nonnumerical data.

System fuzzy interval mathematics is the basis of the modern toolkit of mathematical research methods [5]. It is based on the need to take into account the fuzziness of the initial data and the premises of the mathematical model. Fuzziness is described using fuzzy sets (currently, primarily the first type). The most developed part of the theory of fuzziness is based on membership functions of a special type corresponding to intervals. We are talking about interval mathematics and statistics of interval data. The term "system" has a dual meaning. Firstly, system fuzzy interval mathematics is intended for modeling and analyzing specific systems in various fields of science and sectors of the national economy. Secondly, it emphasizes the need for a systems approach when studying systems of various natures.

One of the options for the practical implementation of system fuzzy interval mathematics is automated system-cognitive analysis and its software implementation: Eidos intelligent system. The scientific results obtained over the first nine years of the development of system fuzzy interval mathematics (2014–2022) are reflected in the monograph [35].

The main ideas of system fuzzy interval mathematics are reflected in scientific and methodological publications. Thus, in the textbook edited by Corresponding Member of the RAS I.I. Eliseeva, it is said: "Sometimes nonparametric econometrics includes econometric analysis of nonnumerical mathematical concepts belonging to certain classes of nonnumerical objects, such as fuzzy sets, intervals, and probability distributions. Thus, in interval data statistics, where the elements of the sample are not numbers, but intervals, almost all problems of classical applied mathematical statistics have been studied, in particular, problems of regression analysis, experimental planning, comparison of alternatives and decision-making under conditions of interval uncertainty, etc. A general research scheme has been developed for this branch of science, including the calculation of two main characteristics—the maximum possible deviation of statistics caused by the interval nature of the initial data, and the rational sample size (exceeding which does not provide a significant increase in the accuracy of estimation and statistical conclusions associated with testing hypotheses). Also, approaches to accounting for interval uncertainty in the main formulations of regression, discriminant, and cluster analysis have been developed" ([36], p. 15). All these scientific results have long been published by the author of this article, including in the textbooks *Econometrics* (2002), and *Applied Statistics* (2006). However, the cited textbook [36] does not contain references to the original sources and does not even mention the name of their author.

## CONCLUSIONS

The main methodological problems of the theory of fuzziness are considered. They should be kept in mind by everyone who develops, applies, and teaches this theory.

It is shown that the history of fuzzy theories begins with the reflections of the philosophers of Ancient Greece. The concept of membership function was introduced by E. Borel. And only after that did the work of L. Zadeh appear, which gave rise to the modern stage of development of the theory of fuzziness. He introduced the operations of the algebra of fuzzy sets and pointed the main road to the "doubling" of mathematics due to the possibility of replacing the numbers and sets of structures of classical areas of mathematics with their fuzzy analogs.

At present, uncertainty can be modeled in three ways—on the basis of probabilistic-statistical models, using the theory of fuzzy sets, and by applying interval mathematics. There are connections between these three methods. The theory of fuzzy sets in a certain sense is reduced to the theory of random sets and thus to probability theory. Intervals are a special case of fuzzy sets in which the membership function is equal to one inside a certain interval and zero outside it.

Statistical analysis of fuzzy data is carried out using methods of statistics of nonnumerical data, primarily using algorithms developed for objects of a general nature.

When discussing the problems of defining terms, it is necessary to take into account that the process of defining some terms through others sooner or later leads to undefined concepts (their analog is axioms in mathematics). The use of linguistic variables (words of natural language) is based on the fact that we think fuzzy and only therefore do we understand each other.

It is important that it is impossible to consistently implement the thesis "Everything in the world is fuzzy." We always rely on clear concepts. This is the main paradox of the theory of fuzziness. For example, for ordinary fuzzy sets (i.e., fuzzy sets of the first order), the values of the argument and the membership function are clear.

A new direction in theoretical and applied mathematics is system fuzzy interval mathematics. It is the basis of the modern toolkit of mathematical research methods. System fuzzy interval mathematics is based on the need to take into account the fuzziness of the initial data and the premises of the mathematical model. It is developed on the basis of a new paradigm of mathematical research methods. Its important components are statistics of nonnumerical data, including statistics of interval data, and the theory of fuzziness. In our opinion, system fuzzy interval mathematics is the basis of mathematics of the 21st century.

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## CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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