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MATHEMATICAL RESEARCH METHODS

Estimating the Parameters of the Gamma Distribution

A. I. Orlov*

Bauman Moscow State Technical University, Moscow, 105005 Russia

**e-mail: prof-orlov@mail.ru*

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Abstract—Formulating problems of statistical analysis of data with a gamma distribution belongs to the classical mathematical statistics domain. Oddly enough, not all problems have been solved within the framework of parametric statistics. These gaps should be bridged, since the gamma distribution is widely used in theoretical and applied research. A plausible approach is provided by the National Standard GOST 11.011–83 Applied statistics. Rules to determine estimates and confidence limits for gamma distribution parameters. The standard gamma distribution is determined by the shape parameter. When switching to the scale-shift family, the scale and shift parameters should be added. Seven formulations of parameter estimation problems are explored, since each of the three parameters can be either unknown or known. For each of the formulations, the estimates of the method of moments and their asymptotic variances are found. The maximum likelihood estimates are obtained for a known shift parameter. Single-step estimates, which are asymptotically equivalent to the maximum likelihood estimates, are used for an unknown shift parameter. The presence of measurement errors affects the accuracy of parameter estimates when certain calculation algorithms are applied. Based on the interval data model, GOST 11.011–83 provides the rules to choose the estimation method for unknown shape and scale parameters when the shift parameter is known. At the stage of developing GOST 11.011–83, problems were identified that required a new approach. The new scientific results obtained in solving practical problems resulted in finding new areas for research. Among those are the statistics of interval data, as well as single-step estimates. The statistics of interval data as a branch of mathematical statistics has reached maturity. It covers all principal areas of statistical methods. It is an important part of systemic fuzzy interval mathematics.

Keywords: statistical methods, gamma distribution, parameter estimation, method of moments, maximum likelihood method, single-step estimates, statistics of interval data, asymptotic distributions, confidence intervals

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INTRODUCTION

Mathematical research methods are based on probability theory and mathematical statistics. The family of gamma distributions is usually mentioned when addressing the issue of continuous probability distributions [1–3]. The methods for estimating the parameters of probability distributions can be applied to this family. Thus, we developed GOST 11.011–83 in the series of Applied Statistics National standards. This standard refers to algorithms for obtaining point estimates and confidence intervals for the parameters of the family and subfamilies of gamma distributions. At the stage of compiling the regulatory and technical norm, a number of research studies were conducted; this resulted in developing sufficiently advanced calculation algorithms in the subject domain. Unfortunately, the standard was cancelled in 1987 along with the entire Applied Statistics series [4]. Past that point, the text of the norm could only be considered as a scientific publication. However, the status of an official regulatory and technical document did not allow one to make this referral. The standard was removed from the standards libraries and was not mentioned in research papers. We feel it is justifiable to note the scientific results that underlie the standard. In fact, these results were subsequently summarized and developed further. We explore them in this paper. For the first time we systematically analyze the problem of estimating the parameters of the gamma distribution.

Formulating tasks dealing with statistical analysis of the data using the gamma distribution belongs to the classical mathematical statistics domain. However, not all problems have been resolved within the framework of parametric statistics, which were at the forefront of statistical science in the first 3 decades of the 20th century. Similar to the case of the beta distribution [5], it is necessary to bridge existing gaps. This is quite topical since the gamma distribution is often used in theoretical and applied work. We can provide the following examples.

Gamma distributions are widely used in various areas of science and practice, specifically in assessing

Table 1. Options to set estimation tasks

Option number	Shape parameter	Scale parameter	Shift parameter
1	To be estimated	Known	Known
2	Known	To be estimated	Known
3	Known	Known	To be estimated
4	Known	To be estimated	To be estimated
5	To be estimated	Known	To be estimated
6	To be estimated	To be estimated	Known
7	To be estimated	To be estimated	To be estimated

reliability features (e.g., in the load-strength model [6]) and in testing theory, in various engineering and applied disciplines (including modeling process accuracy [7]), in meteorology, etc. [8]. It has been established that gamma distributions can be used to model the distribution of the total service life of a product, the length of conductive dust chains, the time it takes for a product to reach its corrosion limit [9], and the time required to develop the k-th failure [10]. In some cases, the life expectancy of patients suffering chronic diseases, the time to achieve a specified treatment effect, and other random variables used in medical and biological research use the gamma distribution. The gamma distribution can be the most appropriate modeling tool in economic and mathematical models of reserve management [11]. It can also be used to map commuter routes [12]. The gamma distribution is widely used in theory and applications (e.g., [13–16]).

The Gamma Distribution Family

The density of the standard gamma distribution can be written as follows:

$$f(x;a) = \begin{cases} \dfrac{1}{\Gamma(a)} x^{a-1} \exp(-x), x \geq 0, \\ 0,\ x < 0, \end{cases} \tag{1}$$

where

$$\Gamma(a) = \int_0^{+\infty} x^{a-1} \exp(-x) \mathrm{d}x \tag{2}$$

is the gamma function.

The standard gamma distribution is defined by a single parameter $a > 0$, which is the shape parameter.

For a random variable X that has a standard gamma distribution the mathematical expectation and the variance equal a. It has the third central moment $M(X - M(X))^3 = 2a$, asymmetry $2/\sqrt{a}$, and excess $6/a$ [1–3].

The standard gamma distribution generates a scale-shift family of random variable distributions

$$Y = bX + c, \tag{3}$$

where b is the scale parameter and c is the shift parameter. As follows from formula (1), the random variable Y given by formula (3) has the density

$$f(x;a,b,c) = \begin{cases} \dfrac{1}{\Gamma(a)} (x-c)^{a-1} b^{-a} \exp\left(-\dfrac{x-c}{b}\right), \\ x \geq c, \\ 0,\ x < c. \end{cases} \tag{4}$$

According to (1) and (4), $f(x; a) = f(x; a, 1, 0)$. Then

$$M(Y) = ab + c, \quad D(Y) = ab^2, \\ M(Y - M(Y))^3 = 2\mathrm{ab}^3. \tag{5}$$

The applicability of the gamma distribution in a number of application domains can be proven by its reproducibility [1–3]. Specifically, the sum of k independent exponentially distributed random variables with the same parameter λ (without a shift) has a gamma distribution with parameters of shape $a = k$, scale $b = 1/\lambda$, and shift $c = 0$. Therefore, the gamma distribution is often used in the same application areas where making use of exponential distribution is appropriate.

Let $Y_1, Y_2, \ldots, Y_n$ be a sample of independent identically distributed random variables that have gamma distribution (4). Here, we address the problem of estimating parameters of this distribution.

The gamma distribution (4) has three parameters: namely, shape, scale, and shift. Each of them can be either known or unknown giving a total of $2^3 = 8$ options. One of these is irrelevant to the tasks at hand, since all parameters are known. To ensure the completeness of the study, it is necessary to consider $2^3 - 1 = 7$ assessment tasks (Table 1).

Estimates of the Method of Moments

The method of moments is the simplest way to estimate distribution parameters [17]. It consists of expressing theoretical moments through distribution parameters (e.g., (5)), followed by solving the inverse problem to find the functional dependence of distribution parameters on theoretical moments. The sample moments are then substituted into the functional dependence instead of the theoretical ones to obtain statistical estimates of the parameters. When the sample size increases, the statistical estimates become asymptotically normal and unbiased. Thus, the linearization method should be used to obtain the asymptotic variances ([17], section 4.4). This approach was implemented for the beta distribution samples [5].

The ambiguity of the method of moments deals with the fact that different sample moments can be used to estimate the same parameter. As an example, in option 1 (see Table 1), we can start with any of the three relationships given in (5), which results in obtaining three estimates for the method of moments:

$$a_1 = a_1(\omega) = \frac{\bar{Y} - c}{b},$$
$$a_2 = a_2(\omega) = \frac{s^2}{b^2}, \quad a_2 = a_2(\omega) = \frac{m_3}{2b^3}, \tag{6}$$

where

$$\bar{Y} = \frac{Y_1 + Y_2 + \ldots + Y_n}{n},$$
$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2, \quad m_3 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^3. \tag{7}$$

Comparing these three estimates (6) can favor the first one since it is the most stable for outliers ([17], section 7.2). As explained above, comparing asymptotic variances of estimates (6) found through linearization brings similar results.

Similarly, for option 2 (see Table 1), we use the estimate

$$b^* = \frac{\bar{Y} - c}{a}, \quad D(b^*) = \frac{1}{n}\frac{b^2}{a},$$

and for option 3 we can make use of the following relationship

$$c^* = \bar{Y} - ab, \quad D(c^*) = \frac{1}{n}ab^2.$$

In options 4–6 we have two unknown parameters; thus, two sampling points should be used. We use the arithmetic mean and the variance. We can start with the following equalities

$$\bar{Y} = ab + c, \quad s^2 = ab^2. \tag{8}$$

In option 4, the scale and shift parameters are estimated based on the sample and the shape:

$$b^* = \frac{s}{\sqrt{a}}, \quad c^* = \bar{Y} - ab^* = \bar{Y} - s\sqrt{a}.$$

Option 5 involves estimating the shape and shift parameters based on the sample and scale:

$$a^* = s^2/b^2, \quad c^* = \bar{Y} - a^*b = \bar{Y} - s^2/b.$$

Option 6 provides for estimating the shape and scale parameters through the sample and shift:

$$\bar{Y} - c = ab, \quad s^2 = ab^2;$$
$$b^* = \frac{s^2}{\bar{Y} - c}, \quad a^* = \frac{\bar{Y} - c}{b^*} = \frac{(\bar{Y} - c)^2}{s^2}.$$

In option 7, all three parameters are unknown, thus three sampling points should be used. We apply the sample arithmetic mean, variance, and the third central moment:

$$\bar{Y} = ab + c, \quad s^2 = ab^2, \quad m_3 = 2ab^3,$$

giving

$$\frac{m_3}{2s^2} = b^*, \quad a^* = \frac{s^2}{(b^*)^2} = \frac{4s^6}{m_3^2},$$
$$c^* = \bar{Y} - a^*b^* = \bar{Y} - \frac{2s^4}{m_3}.$$

The results of evaluating the parameters of the gamma distribution using the method of moments for all seven variants (see Table 1) are given in Table 2. We obtained the asymptotic variances of these estimates using the linearization technique, which was discussed in detail in ([5, 17], section 4.4).

The asymptotic confidence intervals for gamma distribution parameters can be derived based on the normality of the asymptotic estimate. These can be written as follows:

$$\left(\theta_n - C(\gamma)\sqrt{D^*(\theta_n)}; \quad \theta_n + C(\gamma)\sqrt{D^*(\theta_n)}\right),$$

where θ is the considered parameter; θ_n is its estimate (see Table 2); $D^*(\theta_n)$ is the estimate of the θ_n obtained when the unknown parameter values in the formula for the asymptotic variance (rightmost column in Table 2) are replaced by their estimates; and $C(\gamma)$ is the coefficient corresponding to the confidence probability γ (for $\gamma = 0.95$, we have $C(\gamma) = C(0.95) = 1.96$).

Here and below, the confidence intervals are determined based on the asymptotic normality of parameter estimates. For certain samples $Y_1, Y_2, \ldots, Y_n$ calculated using asymptotic formulas, the confidence intervals can go outside the range of definition of the corresponding parameters. In these situations, they should be adjusted. If the lower confidence limit for the shape or scale parameter is negative, it should be replaced with 0. If the upper confidence limit for the shift parameter is greater than any of the sample elements it should be replaced with $\min(Y_1, Y_2, \ldots, Y_n)$. The probabilities of these events tend toward zero as the sample size increases.

The method of moments can be used to test consistency with a family of gamma distributions. Thus, to check consistency with the subfamily of gamma distri-

Table 2. Estimates of the method of moments and their asymptotic variances

Option of setting application problem	Parameter assessed	Type of assessment	Asymptotic estimation variance
1	a	$\frac{\bar{Y} - c}{b}$	$\frac{a}{n}$
2	b	$\frac{\bar{Y} - c}{a}$	$\frac{1}{n}\frac{b^2}{a}$
3	c	$\bar{Y} - ab$	$\frac{1}{n}ab^2$
4	b	$\frac{s}{\sqrt{a}}$	$\frac{b}{2n}(a+3)$
4	c	$\bar{Y} - s\sqrt{a}$	$\frac{ab^2}{2n}(a+1)$
5	a	$\frac{s^2}{b^2}$	$\frac{2a}{n}(a+3)$
5	c	$\bar{Y} - \frac{s^2}{b}$	$\frac{ab^2}{n}(2a+3)$
6	a	$\frac{(\bar{Y} - c)^2}{s^2}$	$\frac{2a(a+1)}{n}$
6	b	$\frac{s^2}{\bar{Y} - c}$	$\frac{b^2}{n}\left(2 + \frac{3}{a}\right)$
7	a	$\frac{4s^6}{m_3^2}$	$\frac{6a}{n}(a^2 + 6a + 5)$
7	b	$\frac{m_3}{2s^2}$	$\frac{b^2}{2an}\left(6a^2 + 25a + 24\right)$
7	c	$\bar{Y} - \frac{2s^4}{m_3}$	$\frac{ab^2}{2an}\left(3a^2 + 13a + 10\right)$

butions $f(x; a, b, 0)$ with shift parameter $c = 0$, we can use the criterion with the following statistics

$$Z = \sqrt{n}\left(\frac{\bar{Y}m_3}{2s^4} - 1\right).$$

Using the method described above to find the asymptotic distribution of a function based on the sample moments, we obtain that as the sample size increases, the distribution of Z converges to normal and if the hypothesis that the shift parameter equals zero is correct the limiting Z distribution has a zero mathematical expectation and the variance can be expressed through

$$\frac{1}{2a}(3a^2 + 13a + 10).$$

Since the parameter a is unknown, its consistent estimate a^* should be substituted into the expression for variance. The statistical criterion for testing the null hypothesis $c = 0$ under alternative $c > 0$ has the critical domain

$$\left\{Z : Z > u(1 - a)\sqrt{\frac{3(a^*)^2 + 13a^* + 10}{2a^*}},\right\}$$

where $u(1 - \alpha)$ is the quantile of the order $1 - \alpha$ of the standard normal distribution that has a mathematical expectation of 0 and a variance of 1. As $n \to \infty$, the significance level of the criterion approaches α. For more details on applying the method of moments to test the hypothesis of agreement with a parametric family of distributions, refer to ([17], section 7.1).

Maximum Likelihood Estimates (with Known Shift Pasrameter)

Moment estimates are the simplest type of parameter estimates. In many cases, however, they are not the best asymptotically normal (ANE) estimates because their variance is greater than that provided by other techniques. The maximum likelihood estimates are the most popular of the ANE estimates. They are obtained by maximizing the parameters of the product of densities, in which the elements of the sample are substituted for arguments. In the case of a gamma distribution with density (4), we are maximizing the likelihood function.

$$\begin{aligned} F &= \prod_{1\le i\le n} f(Y_i; a, b, c) \\ &= \prod_{1\le i\le n} \frac{1}{\Gamma(a)}(Y_i - c)^{a-1} b^{-a} \exp\left(-\frac{Y_i - c}{b}\right), \end{aligned} \tag{9}$$

using parameters $a > 0$, $b > 0$, $c < \min(Y_1, Y_1, \dots, Y_n)$.

It is easier to deal with a logarithmic likelihood function

$$\begin{aligned} L(a,b,c) &= \ln F = -n\ln\Gamma(a) + (a-1) \\ &\times \sum_{i=1}^{n}(\ln(Y_i - c) - an\ln b - \frac{1}{b}\sum_{i=1}^{n} Y_i + \frac{nc}{b}. \end{aligned} \tag{10}$$

For setting option 1, the value of the parameter a can be found from the equation

$$\frac{\partial L}{\partial a} = -n\frac{\mathrm{d}\ln\Gamma(a)}{\mathrm{d}a} + \sum_{i=1}^{n}\ln(Y_i - c) - n\ln b = 0. \tag{11}$$

Let us introduce the logarithmic derivative of the gamma function

$$\Psi(a) = \frac{\mathrm{d}\ln\Gamma(a)}{\mathrm{d}a} = \frac{1}{\Gamma(a)}\frac{\mathrm{d}\Gamma(a)}{\mathrm{d}a}$$

this special function is known as the psi function [18]). From (11) it follows that the estimate of the shape parameter under known scale and shift parameters is found as the solution of the equation

$$\Psi(a) = \frac{1}{n}\sum_{i=1}^{n}\ln\left(\frac{Y_i - c}{b}\right),$$

i.e., the asymptotically normal estimate of the shape parameter has the form

$$a^* = G\left(\frac{1}{n}\sum_{i=1}^{n}\ln\left(\frac{Y_i - c}{b}\right)\right),$$

where G is the expression that is the inverse of the psi function. Section 4 of GOST 11.011–83 provides rules to calculate the G function, the dispersion of the estimate a^*, and the confidence limits based on using tables and approximate formulas. Currently, it is advised to use the appropriate software.

For setting option 2, the evaluation of the parameter b^* is given by the solution to the equation

$$\begin{aligned} \frac{\partial L}{\partial b} &= \frac{\partial L}{\partial b}\left(-an\ln b - \frac{1}{b}\sum_{i=1}^{n}(Y_i - c)\right) \\ &= -\frac{an}{b} + \frac{1}{b^2}\sum_{i=1}^{n}(Y_i - c) = 0, \end{aligned} \tag{12}$$

i.e.,

$$b^* = \frac{1}{na}\sum_{i=1}^{n}(Y_i - c). \tag{13}$$

The tables and approximate (asymptotic) formulas to calculate confidence limits for the parameter b are given in section 3 of GOST 11.011–83. At the current stage of developed applied statistics, software packages are used for these purposes.

Let us evaluate the parameters of shape and scale when the shift parameter is known (setting option 6). The maximum likelihood estimates are found by solving the system of equations (11)–(12). The scale parameter is evaluated using formula (13). Substituting it into (11), we obtain the equation for the shape parameter:

$$\begin{aligned} &-n\frac{\mathrm{d}\ln\Gamma(a)}{\mathrm{d}a} + \sum_{i=1}^{n}\ln(Y_i - c) \\ &-n\ln\left(\frac{1}{n}\sum_{i=1}^{n}(Y_i - c)\right) + n\ln a = 0. \end{aligned}$$

Let us transform this equation:

$$\ln a - \Psi(a) = \ln\left(\frac{1}{n}\sum_{i=1}^{n}(Y_i - c)\right) - \frac{1}{n}\sum_{i=1}^{n}\ln(Y_i - c).$$

Consequently, the estimates of the shape and scale parameters for a known shift parameter are as follows:

$$\begin{aligned} a^* &= H\left[\ln\left(\frac{1}{n}\sum_{i=1}^{n}(Y_i - c)\right) - \frac{1}{n}\sum_{i=1}^{n}\ln(Y_i - c)\right], \\ b^* &= \frac{1}{na^*}\sum_{i=1}^{n}(Y_i - c), \end{aligned} \tag{14}$$

where H is the inverse function of the function

$$Q(a) = \ln(a) - \Psi(a).$$

GOST 11.011–83 (Section 7) outlines algorithms for calculating estimates (14), their variances, and confidence intervals based on the relevant tables and approximate (asymptotic) formulas developed at the stage of compiling the standard. In a similar manner to the previously discussed setting options, the results obtained in applied mathematics can now be used in compiling appropriate software.

Using the linearization method (see also [19], pp. 83, 98), we can show that for large n, with accuracy to infinitesimals of higher order

$$M(a^* - a)^2 = \frac{a}{n(aI(a) - 1)},$$

$$M(b^* - b)^2 = \frac{b^2 I(a)}{n(aI(a) - 1)}, \quad I(a) = \frac{\mathrm{d}\Psi(a)}{\mathrm{d}(a)}.$$

Based on the properties of the gamma function [19], GOST 11.011–83 establishes that for large a

$$M(a^* - a)^2 = \frac{a(2a - 1)}{n}, \quad M(b^* - b)^2 = \frac{2b^2}{n},$$

with infinite accuracy of a higher order. Using Table 2, we conclude that the mean squares of errors for the estimates of the method of moments for the shape parameter a^{**} and the scale parameter b^{**} (under the known shift parameter)

$$M(a^{**} - a)^2 = \frac{2a(a + 1)}{n},$$

$$M(b^{**} - b)^2 = \frac{b^2}{n}\left(2 + \frac{3}{a}\right),$$

is greater than the relevant mean squares of errors found for the maximum likelihood estimates. Thus, for setting option 6, the maximum likelihood estimates prevail over moment estimates. As established in classical mathematical statistics theory, this is the case with the most problems in parameter estimation. Maximum likelihood estimates are always the best asymptotic normal estimates, whereas moment estimates are only better in certain cases.

Singsle-Step Estimates
(When the Shift Parameter Is Unknown)

In setting options 3, 4, 5, and 7, the shift parameter c is unknown and should be estimated.

The maximum likelihood estimate c^* of parameter c in option 3 can be determined from the condition

$$\frac{\partial L}{\partial c} = \frac{\partial L}{\partial c}\left[(a - 1)\sum_{i=1}^{n} \ln(Y_i - c) + \frac{nc}{b}\right] = -(a - 1)\sum_{i=1}^{n} \frac{1}{Y_i - c} + \frac{n}{b} = 0. \tag{15}$$

In option 4, the estimates of the scale and shift parameters are given by solutions of the maximum likelihood equations (12) and (15). Substituting the expression for estimating the scale parameter (13) into (15), we obtain an estimate of the shift parameter c^* from the equation

$$-\frac{a - 1}{n}\sum_{i=1}^{n} \frac{1}{Y_i - c} + \frac{a}{\frac{1}{n}\sum_{i=1}^{n}(Y_i - c)} = 0. \tag{16}$$

In option 5, the shape and shift parameters are found through solving the system of maximum likelihood equations (11) and (15). Substituting the solution of equation (11) into (15), we obtain the equation to estimate the shift parameter

$$\left\{G\left(\frac{1}{n}\sum_{i=1}^{n} \ln\frac{Y_i - c}{b}\right) - 1\right\} - \frac{n}{b} = 0, \tag{17}$$

where G is the function inverse of the psi function. The calculation rules are specified in GOST 11.011–83.

Under three unknown parameters (option 7), the maximum likelihood estimates should be found by solving a system of three equations: (11), (12), and (15).

Thus, in a number of settings, in order to obtain maximum likelihood estimates one should solve the corresponding equations or systems of equations. It would seem reasonable that it would suffice to apply common numerical techniques. However, this approach brings a number of problems.

In many parameter estimation problems, there are no available algorithms based on explicit formulas to find maximum likelihood estimates; therefore, numerical methods should be used. This is true not only for the gamma distribution, but also, for example, for the Weibull–Gnedeko distribution. Then, the system of maximum likelihood equations is solved using an appropriate iterative (see, for example, [20]) or the likelihood function is maximized directly (see, for example, [21]).

However, in contrast to algorithms based on explicit formulas (as is the case for the normal distribution) the application of numerical methods faces numerous hurdles. Thus, the convergence of iterative methods should be justified. In a number of examples, the likelihood function has many local maxima; therefore, the natural iterative procedures do not converge [22]. As an example, for data on fatigue steel testing, the maximum likelihood equation has as many as 11 roots [23]. It is not clear which one should be used as the parameter estimate. To solve the problem of convergence of iterative methods, research should be carried out to prove the convergence of iterative algorithms to find the maximum likelihood estimates for specific probability models and for specific algorithms (see, for example, [24]).

We should point out that mathematical proof of the convergence of an iterative algorithm is not sufficient. The important issue is choosing the right moment to stop calculations once the required accuracy is achieved. In most cases, the issue remains open. In addition, the accuracy of calculations should be associated with the sample size: the larger it is, the more accurate the parameter estimates should be. Otherwise, the estimation method cannot be considered reliable. It is important to note that as the sample size increases, the algorithms should be made more complex, in particular by increasing the number of significant digits in computer calculations, shifting from the single to double accuracy estimates, and so on, in order to achieve reliable results.

Thus, if there are no clear formulas for maximum likelihood estimates, finding them runs into a number of computational problems. Mathematical statisticians usually ignore all these problems when discussing maximum likelihood estimates in theoretical terms. However, for applied statistics this is crucial, since any doubt should be avoided about the applicability of maximum likelihood estimates.

In addition to the method of moments and maximum likelihood estimates, applied statistics developed other approaches. As an example, quantile estimates are based on an idea similar to the method of moments, but instead of sample and theoretical moments, the sample and theoretical quantiles are equated. Another group of estimates is based on the idea of minimizing (in terms of distribution parameters) a certain difference (difference indicator) between empirical data and an element of a parametric family. As an example, the Euclidean or block difference between empirical and theoretical histograms is minimized. (To be more precise, theoretical histograms are composed of the vectors generated by the heights of histogram bars.)

There is no need to focus only on maximum likelihood estimates. There are other types of estimates that have similar statistical properties. As an example, single-step estimates are the result of the first iteration of the Newton–Raphson method when solving a maximum likelihood equation or system of equations. These estimates are the best among the normal asymptotic estimates, i.e., they have similar asymptotic properties as the maximum likelihood estimates under the same mathematical conditions of regularity. Single-step estimates are obtained in the form of explicit formulas, i.e., these estimates are free of the computational problems discussed above. In their calculations we can take the moment method estimate as an initial approximation for finding corrections based on the first iteration of the Newton–Raphson technique.

For the gamma distribution in scenario 3, it is recommended to use a single-step estimate instead of solving equation (15). The estimate of the method of moments $\overline{Y} - ab$ is taken as an initial approximation. The calculation formulas are given in GOST 11.011–83, sections 8.15–8.18. For options 4 and 7, the rules to determine single-step estimates are provided in chapter 8.

The development of GOST 11.011–83 stimulated further development of the theory of single-step estimates. This theory is described in detail in textbooks ([17], section 6.2; 25, section 3.25) that address its application for estimating gamma distribution parameters. Therefore, we will not dwell on the above scientific results in this paper.

The Statistics of Interval Data

The development of GOST 11.011–83 laid the foundations for interval data statistics, which has become a solid area in modern mathematical statistics. The interest in this topic was triggered by the analysis of data on the limiting wear of cutting tools (in hours). This area of mathematical statistics was also covered in subsequent publications (see, for example, [17], section 6.1). It is quite obvious that all initial data are specified either as natural numbers or as sums of natural numbers and 0.5 (for example, 127.5 hours). This implies that the wear of the cutters to their limit state is measured with an accuracy of 0.5 hour. Obviously, the presence of errors should be taken into account when processing the data. When developing GOST 11.011–83, it was agreed that

$$Z_i = Y_i + \varepsilon_i, \quad i = 1, 2, \ldots, n, \quad |\varepsilon_i| \le \Delta, \tag{18}$$

where n is the sample size; Z_i is the statistical data to be analyzed; Y_i are real independent random variables that have a gamma distribution; ε_i are independent measurement errors (inaccuracies); and Δ is the maximum error.

Measurement errors affect the accuracy of parameter estimates when certain calculation algorithms are applied. Based on an interval data model (18), GOST 11.011–83 (see section 5) provides rules to choose an estimation technique for unknown shape and scale parameters when the shift parameter is known.

Further developments in interval data statistics are given in numerous papers (see, for example, review [26]). This branch of mathematical statistics is quite well developed and covers all main areas of statistical methods. It is an important component of systemic fuzzy interval mathematics ([27], chapter 7) and it is described in detail in a number of our textbooks (see, for example, [17], chapter 12).

CONCLUSIONS

Information is presented on the main results from developing methods to estimate gamma distribution parameters based on limit theorems of probability theory and mathematical statistics. The next step is to build and study evaluation algorithms for samples of finite size.

It should be noted that, in a similar manner to maximum likelihood estimates, single-step estimates are asymptotically optimal. However, in some cases and for specific sample sizes unbiased estimates can prove to be more accurate [28].

It is interesting to trace the development of research in this area from the historical perspective. When solving a practical problem (i.e., when developing GOST 11.011–83), problems were identified that required new scientifically justified methods. New scientific results obtained in solving a practical problem resulted in forming new scientific areas. Among the latter are interval data statistics and single-step estimates. Naturally, the proposed theory found practical applications.

This scenario is not unique. Another example deals with developing the ASFPAI (Automated System to Forecast and Prevent Aviation Incidents). As mentioned in [29], development of the system was feasible based on the large scope of specific scientific research. Some of these involve modeling aviation incidents at various stages of flight, which involves analyzing as many as 12 event trees (directed graphs) that have specific gear ratios when progressing from the input data to the tops of the trees. New expert assessment procedures have been developed and implemented. New methods for short- and long-term forecasting, assessment, and risk management have been proposed. The ASFPAI significantly reduces the risk of aviation incidents. As many as 12 areas were suggested in [29] for further development of the ASFPAI. Research in these areas is ongoing.

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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